Matrix Difference Equation Analysis of Vibrating Periodic Structures

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The Matrix Difference Equation (MDE) method for sound transmission and forced vibration analysis of damped periodic structures is presented. A periodic structure is defined as a string of identical substructures, such as a segment of aircraft fuselage which has identical bays between circumferential frames. The finite element method is applied to the substructure to provide a mechanical impedance matrix. A matrix difference equation is derived from the impedance matrix, based upon conditions of equilibrium and compatibility at substructure boundaries. The difference equation is reduced in order by eliminating force variables and introducing substructure displacement modes. A solution is found by calculating eigenvalues and eigenvectors of a related characteristic equation. The result is a closed form expression in the longitudinal coordinate. The method is general and applicable to complex structures, because of the finite element basis. Results of an application to aircraft engine duct vibration are included.

Nomenclature

A = mechanical impedance matrix (square)
A_B = boundary mechanical impedance matrix (square)
$\hat{A}_B$ = transformed boundary mechanical impedance matrix
$A_{Bll}, A_{Blr}, A_{Brl}, A_{Brr}$ = partitions of A_B corresponding to left- and right-hand substructure boundaries
$\hat{A}_{Bll}, \hat{A}_{Blr}, \hat{A}_{Brl}, \hat{A}_{Brr}$ = partitions of $\hat{A}_B$
C = a column matrix of arbitrary constants needed to satisfy the boundary conditions. Also the substructure damping matrix (square)
C_B = the substructure boundary damping matrix (square)
C_k = kth element of matrix of arbitrary constants C
C_s = column matrix of arbitrary constants for symmetric loads
e = base of natural logarithms
$F^{(i)}$ = column matrix of eigenvalues $\hat{\lambda}_k$ raised to the ith power
$\hat{G}_k$ = a response eigenvector (column matrix) for the transformed structure
$\hat{G}_{P_k}, \hat{G}_{\Delta_k}$ = column matrices of load and displacement eigenvectors for the transformed structure
$\hat{G}_{\Delta a_k}, \hat{G}_{\Delta b_k}$ = column matrices of a-type and b-type displacement eigenvectors for the transformed structure
$\hat{H}$ = a square matrix of response eigenvectors
$\hat{H}_P, \hat{H}_\Delta$ = rectangular matrices of load and displacement eigenvectors
$\hat{H}_{Pa}, \hat{H}_{Pb}$ = rectangular matrices of a-type and b-type load eigenvectors
$\hat{H}_{\Delta a}, \hat{H}_{\Delta b}$ = rectangular matrices of a-type and b-type displacement eigenvectors
j = $(-1)^{1/2}$
K = substructure stiffness matrix (square)
K_B = substructure boundary stiffness matrix (square)
M = substructure mass matrix (square)
M_B = substructure boundary mass matrix (square)
$\hat{n}$ = number of transformed boundary degrees of freedom
$\hat{n}_a, \hat{n}_b$ = numbers of transformed a-type and b-type boundary degrees of freedom
P_U = column matrix of substructure unconstrained loads
P_{UB} = column matrix of a substructure unconstrained boundary loads
$\hat{P}_{UB}$ = column matrix of substructure transformed unconstrained boundary loads
P_{UBl}, P_{UBr} = column matrices of loads on the left and right boundaries
$\hat{P}_{UBl}, \hat{P}_{UBr}$ = column matrices of transformed loads on the left and right boundaries
P_{UBla}, P_{UBra} = column matrices of a-type loads on the left and right boundaries
$\hat{r}_k$ = modulus of the eigenvalue $\hat{\lambda}_k$
t = time
T = rectangular transformation matrix
$\hat{Y}_k^{(i)}$ = column matrix of transformed load and displacement responses at the ith substructure boundary corresponding to $\hat{\lambda}_k$
$\alpha_{aa}, \alpha_{ab}, \alpha_{bb}$ = partitions of A_{Bll} and A_{Brr}
Δ = column matrix of substructure displacements
Δ_B = column matrix of substructure boundary displacements
Δ_{Bl}, Δ_{Br} = column matrices of left and right boundary displacements
$\hat{\Delta}_{Bl}, \hat{\Delta}_{Br}$ = column matrices of transformed left and right boundary displacements
$\hat{\Delta}_{Bra}, \hat{\Delta}_{Brb}$ column matrices of transformed a-type and b-type displacements of the right-hand boundary
$\hat{\theta}_k$ = argument of the eigenvalue $\hat{\lambda}_k$
$\hat{\lambda}_k$ = kth eigenvalue of the governing characteristic equation for the transformed structure
$\hat{\lambda}_D$ = diagonal matrix of the eigenvalues $\hat{\lambda}_k$
$\hat{\lambda}_{R_k}, \hat{\lambda}_{I_k}$ = real and imaginary parts of $\hat{\lambda}_k$
$\bar{\lambda}_k$ = kth eigenvalue of the governing characteristic equation after force variables and b-type displacement variables have been eliminated
λ_k^* = $(\hat{\lambda}_k - 1/\hat{\lambda}_k)/2$
λ_D^* = diagonal matrix of the λ_k^*'s
$\mu_{aa}, \mu_{ab}, \mu_{bb}$ = partitions of A_{Brl} and A_{Blr}
Φ_a = column matrix of complex amplitudes of a-type loads externally applied at $i = 0$
$\hat{\Phi}_a$ = column matrix of complex amplitudes of transformed a-type loads externally applied at $i = 0$

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Introduction

THE Matrix Difference Equation (MDE) method is designed to facilitate sound transmission and vibration analyses of periodic structures. The term "periodic structure," in this context, means a structure composed of identical substructures joined along a longitudinal axis. An example of such a structure is a segment of aircraft fuselage in which each bay between circumferential frames is identical with every other bay. The method is based upon the finite element approach, and the substructures considered are finite element models.

The principle of the MDE method is that substructures are joined mathematically by writing and solving a matrix difference equation expressing equilibrium and compatibility at substructure boundaries. Consequently, the analysis of the joined structure can be accomplished with approximately the effort necessary to analyze a single substructure, regardless of length of structure. The MDE method therefore represents a significant improvement in the capability of analyzing periodic structures, in respect to cost and speed of calculation, compared to the standard finite element approach. Because of the finite element basis, the method is applicable to periodic structures in general. Highly complex configurations can be analyzed.

Many sound transmission and vibration problems occur in the aerospace industry which require the realism of the finite element approach. Yet such a treatment is not feasible, because the computing effort involved in a damped vibration analysis of a truly realistic model can be prohibitive. In particular, the computing effort increases rapidly with length of model, and it is important that this length be adequate to minimize the effects of extraneous natural modes and frequencies introduced at boundaries unrealistically close to points of application of exciting forces, and also to represent properly the damping effect of structure remote from these points. In such cases the MDE method can be of maximum utility, because the computing effort required is essentially independent of model length, and even infinitely long models can be analyzed.

The assumption of periodicity may seem restrictive, but many aerospace structures are designed to have this property to minimize costs. Furthermore, moderate departures from periodicity can be expected to have negligible effects on results, if the periodic model represents the area of interest of the real structure.

Considerable literature is devoted to the analysis of periodic structures. Reference 1, published in 1940, contains an analysis of a building frame, for which a difference equation is written and solved. Donaldson and Lin[2] describe the transfer matrix technique for periodic structures and give examples of applications to beam and aircraft panel vibration. The technique involves multiplication of a chain of transfer matrices. According to Lin and McDaniel,[3] this approach has limitations because accuracy is lost in the calculations. Reference 3 also mentions a difference equation technique, but describes a "complementary approach" based on the Cayley-Hamilton theorem. Sen Gupta[4] describes the analysis of vibrating periodic structures (beams and plates) by a "wave approach" that appears to be similar in some respects to the methods presented in this paper. Mead[5] writes a difference equation for the response of a damped uniform beam on regularly spaced supports. The equation is solved in a manner equivalent in principle to the methods of the present paper, but the structure considered in the reference is very simple, and is governed by a scalar equation, rather than a matrix equation. Mead and Sen Gupta[6] apply a similar approach to a class of damped rib-skin structures. This class of structures is again limited.

Mead[7] applies the difference equation technique to a general class of structures. The approach is similar to the present method, but significant differences exist. Equation (8a) of the reference is a characteristic equation which is the same as Eq. (23) of the present paper, except that no allowance is made for a transformation of the variable to reduce the order of the matrices involved, and off-diagonal partitions of the mass matrix are set to zero. However the most significant difference lies in the method of solving for eigenvalues and eigenvectors. The characteristic equation is quadratic in the eigenvalue. Such an equation cannot be solved by standard linear methods without doubling the order of the matrices involved. This approach would increase the required computing effort by a large factor. Instead, Mead expands the determinant of the equations as a means of calculating eigenvalues. The resulting computing task is formidable for practical problems. In the present paper the fact is recognized that two catagories of displacement variables exist, designated "a" and "b," corresponding to displacement vectors that are symmetric and antisymmetric about the substructure plane of symmetry. The b-type variables are eliminated. The result is a linear characteristic equation of the same order as the original quadratic equation. The linear equation can be solved for eigenvalues and eigenvectors by existing highly developed computer methods. This approach has the further advantage of providing added insight into the problem.

Reference 8 provides a review of research on high-frequency vibration of aircraft structures. This reference contains an extensive bibliography. Cockburn and Jolly present an analysis of an accoustically excited circular cylindrical shell, and some other models, based on a differential equation approach.[9] The approach is analogous to the difference equation solution, but the latter has greater generality. Bushnell has contributed extensively to the field of shell stress, stability, and vibration analysis.[10] His work is applicable to shells of revolution, and does not take advantage of structural periodicity, as in the present approach.

In the following section the difference equation approach is cast in general form and combined with the finite element method to provide a technique applicable to periodic structures in general. The solution to the difference equation is given in terms of eigenvalues and eigenvectors of a matrix derived from the substructure mechanical impedance matrix. An important feature of the approach is that force variables are eliminated from the difference equation, so that the order of the linear characteristic equation to be solved for eigenvalues and eigenvectors is reduced by a factor of two. This feature is important because the computational effort required to solve a characterstic equation increases rapidly with the order of the equation. Another feature of the method is the introduction of a modal technique that further reduces the order of the characteristic equation. Novel features of this approach include the following: 1) The generality of the formulation incorporating finite element analysis procedures. 2) Elimination of force variables from the difference equations, based upon the discovery that boundary degrees of freedom fall into two categories.

Responses calculated by the MDE method can include detailed stress distributions; consequently, the method is applicable to the field of acoustic fatigue. If a fictitious force or incremental stiffness matrix is substituted for the mass matrix, the MDE approach becomes applicable to the structural stability problem. The static stress problem also can be solved, but the general stability and static stress problems require the addition of certain "algebraic modes" in addition to the "exponential modes" that appear in the vibration solution.

Matrix Difference Equation Method

The basic equations of the MDE method are derived.

Assumptions

The structure is spatially periodic and linear. Additional considerations involved in the assumption of periodicity are presented in the next paragraph. The assumption of linearity implies material linearity and small deflections, so that structural responses are linear functions of applied loads. The MDE method is applicable to structures of either finite or infinite length. The solution for the governing difference equation given subsequently is general, but it contains arbitrary constants that must be evaluated to satisfy the boundary conditions. Because of space limitations, these constants are evaluated only for the case of infinitely long structures.

Periodic Structure

Figure 1 shows a finite element model of the structure under consideration, consisting of substructures joined along a longi-

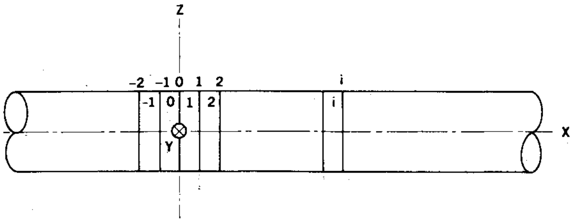


Fig. 1 The structure.

tudinal axis. Substructures and boundaries are numbered as shown. The ith boundary is the right-hand boundary of the ith substructure. All substructures are identical, and each substructure is symmetric about a median plane normal to the longitudinal axis.

Substructure

Figure 2 shows a typical substructure, composed of finite elements. The substructure can have reactions, as at a longitudinal plane of symmetry. However, the substructure need not have reactions if it includes the entire cross section of the structure, as in Fig. 2. Degrees of freedom can be translational or rotational. Degrees of freedom on a boundary are either normal to the boundary or in the boundary plane. Degrees of freedom on both boundaries of the substructure must correspond.

Equation of Motion

Stiffness, mass, and damping matrices are derived for the substructure by standard finite element methods. The linear equation of motion for the substructure in harmonic motion is

$$A\Delta = P_U \tag{1}$$

where A is the mechanical impedance matrix, Δ is a column matrix of complex displacement amplitudes, and P_U is a column matrix of complex external load amplitudes in the unconstrained degrees of freedom. The impedance matrix is given by

$$A = K - \omega^2 M + j\omega C \tag{2}$$

where ω = frequency (rad/sec), $j = (-1)^{1/2}$ and K, M, and C = square stiffness, mass, and damping matrices for the substructure.

The damping matrix C should reflect the presence of member damping, added damping treatments if any, and acoustic radiation damping (important). The real and imaginary parts of Δ and P_U can be interpreted as column matrices of displacements and loads in the unconstrained degrees of freedom at time $t = 0$ and $t = -\pi/(2\omega)$. Note the similarity between Eq. (1) and the displacement equation for static analysis: $K\Delta = P_U$. The mechanical impedance matrix can be regarded as a complex stiffness matrix.

Boundary Impedance Matrix

At this point assume that substructure interior degrees of freedom are eliminated. Thus interior loads can be replaced by equivalent boundary loads, and an equivalent boundary stiffness matrix can be found in terms of interior and boundary stiffnesses, by standard procedures. Procedures are also available for replacing mass and damping matrices by equivalent boundary matrices. As a result

$$A_B \Delta_B = P_{UB} \tag{3}$$

where

$$A_B = K_B - \omega^2 M_B + j\omega C_B \tag{4}$$

Δ_B, P_{UB} = colum matrices of complex amplitudes of displacements and equivalent loads in the boundary degrees of freedom.

K_B, M_B, C_B = equivalent square boundary substructure stiffness, mass, and damping matrices.

Modal Matrix Difference Equation (MMDE) Method

The matrix difference equation is derived and solved.

Modal Transformation

We assume a transformation of the form

$$\Delta_B = T\hat{\Delta}_B \tag{5}$$

where $\hat{\Delta}_B$ has fewer elements than Δ_B, so that the magnitude of the subsequent computation is reduced. Applying this transformation to Eq. (3) gives

$$\hat{A}_B \hat{\Delta}_B = \hat{P}_{UB} \tag{6}$$

where

$$\left.\begin{aligned} \hat{A}_B &= T^T A_B T \\ \hat{P}_{UB} &= T^T P_{UB} \end{aligned}\right\} \tag{7}$$

Let $\hat{n}$ = number of transformed degrees of freedom per boundary. The order of $\hat{A}_B$ is $2\hat{n}$. The impedance, displacement, and load matrices can be partitioned as follows:

$$\begin{bmatrix} A_{Bll} & \vdots & A_{Blr} \\ \cdots & \cdots & \cdots \\ A_{Brl} & \vdots & A_{Brr} \end{bmatrix} \begin{Bmatrix} \Delta_{Bl} \\ \Delta_{Br} \end{Bmatrix} = \begin{Bmatrix} P_{UBl} \\ P_{UBr} \end{Bmatrix} \tag{8}$$

where the subscripts l and r refer to the left and right boundaries of the substructure. For example, Δ_{Bl} is a column matrix of displacements of the left-hand substructure boundary (see Fig. 2). Applying the transformation T to Eq. (8) gives an equation of similar form, where the partitions of $\hat{\Delta}_B$, $\hat{P}_{UB}$ and $\hat{A}_B$ are denoted $\hat{\Delta}_{Bl}$, $\hat{P}_{UBl}$ $\hat{A}_{Bll}$, etc.

Force Variables Retained

The difference equation can be derived either by retaining or eliminating the force matrix P_U. Retention of the force variables leads to large matrices in subsequent calculations, and the resulting procedure is less useful and provides less insight into the problem than the procedure resulting from elimination of the force variables. However, retaining the force variables provides a procedure that is easier to understand. Both approaches are presented. The transformed form of Eq. (8) can be rewritten

$$\left.\begin{aligned} \hat{A}_{Bll}\hat{\Delta}_{Bl}^{(i)} + \hat{A}_{Blr}\hat{\Delta}_{Br}^{(i)} &= \hat{P}_{UBl}^{(i)} \\ \hat{A}_{Brl}\hat{\Delta}_{Bl}^{(i)} + \hat{A}_{Brr}\hat{\Delta}_{Br}^{(i)} &= \hat{P}_{UBr}^{(i)} \end{aligned}\right\} \tag{9}$$

The superscripts "i" indicate that the equations apply to the ith substructure. No superscripts are attached to $\hat{A}_{Bll}$ etc., because these quantities are the same for all substructures. Now because of the continuity of the structure we can write

$$\left.\begin{aligned} \hat{\Delta}_{Bl}^{(i)} &= \hat{\Delta}_{Br}^{(i-1)} \\ \hat{P}_{UBl}^{(i)} &= -\hat{P}_{UBr}^{(i-1)} \end{aligned}\right\} \tag{10}$$

Eliminating $\hat{\Delta}_{Bl}^{(i)}$ and $\hat{P}_{UBl}^{(i)}$ from Eqs. (9) and (10), and solving for $\hat{\Delta}_{Br}^{(i)}$ and $\hat{P}_{UBr}^{(i)}$ gives

$$\hat{Y}^{(i+1)} - \hat{N}\hat{Y}^{(i)} = 0 \tag{11}$$

where

$$\hat{N} = \begin{bmatrix} -\hat{A}_{Brr}\hat{A}_{Blr}^{-1} & \vdots & \hat{A}_{Brl} - \hat{A}_{Brr}\hat{A}_{Blr}^{-1}\hat{A}_{Bll} \\ \cdots & \cdots & \cdots \\ -\hat{A}_{Blr}^{-1} & \vdots & -\hat{A}_{Blr}^{-1}\hat{A}_{Bll} \end{bmatrix} \quad \hat{Y}^{(i)} = \begin{Bmatrix} \hat{P}_{UBr}^{(i)} \\ \hat{\Delta}_{Br}^{(i)} \end{Bmatrix} \tag{12}$$

Equation (11) is a matrix difference equation. A solution is

$$\hat{Y}_k^{(i)} = \hat{G}_k \hat{\lambda}_k^i \tag{13}$$

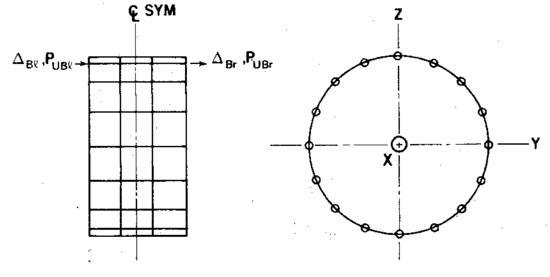


Fig. 2 A substructure.

where $\hat{G}_k$ is a column matrix and $\hat{\lambda}_k$ is a scalar. Substituting $\hat{Y}_k^{(i)}$ from Eq. (13) into Eq. (11) gives

$$(\hat{N}-\hat{\lambda}_k I)\hat{G}_k = 0 \tag{14}$$

Equation (14) is a characteristic equation which can be solved for the eigenvalues $\hat{\lambda}_k$ and the eigenvectors $\hat{G}_k$. The range of the subscript k is equal to $2\hat{n}$, the order of $\hat{N}$. The general solution of the problem is

$$\hat{Y}^{(i)} = \sum_k C_k \hat{\lambda}_k^i \hat{G}_k \tag{15}$$

where the C_k's are arbitrary constants determined by the boundary conditions. Now

$$\hat{\lambda}_k = \hat{\lambda}_{R_k} + j\hat{\lambda}_{I_k} = \hat{r}_k \exp(j\hat{\theta}_k) \tag{16}$$

where $\hat{r}_k$ and $\hat{\theta}_k$ are the modulus and argument of $\hat{\lambda}_k$.

$$\therefore \quad \hat{Y}^{(i)} = \sum_k C_k \hat{G}_k \hat{r}_k^i (\cos\hat{\theta}_k i + j\sin\hat{\theta}_k i) \tag{17}$$

Equation (15) can be written in the form

$$\hat{Y}^{(i)} = \hat{H} C_D F^{(i)} \tag{18}$$

where $\hat{H}$ is a square matrix of eigenvectors $\hat{G}_k$, C_D is a diagonal matrix of arbitrary constants C_k, and $F^{(i)}$ is a column matrix of eigenvalues $\hat{\lambda}_k$, each raised to the ith power.

The formulation with force variables retained is useful for solving small problems involving limited numbers of degrees of freedom.

Force Variables Eliminated

Eliminating $\hat{\Delta}_{Bl}^{(i)}$ and $\hat{P}_{UBl}^{(i)}$ from Eqs. (9) and (10) gives

$$\hat{A}_{Bll}\hat{\Delta}_{Br}^{(i-1)} + \hat{A}_{Blr}\hat{\Delta}_{Br}^{(i)} = -\hat{P}_{UBr}^{(i-1)} \tag{19}$$

$$\hat{A}_{Brl}\hat{\Delta}_{Br}^{(i-1)} + \hat{A}_{Brr}\hat{\Delta}_{Br}^{(i)} = \hat{P}_{UBr}^{(i)} \tag{20}$$

Increasing the superscripts in Eq. (19) by unity and adding the result to Eq. (20) leads to

$$\hat{A}_{Brl}\hat{\Delta}_{Br}^{(i)} + (\hat{A}_{Bll}+\hat{A}_{Brr})\hat{\Delta}_{Br}^{(i+1)} + \hat{A}_{Blr}\hat{\Delta}_{Br}^{(i+2)} = 0 \tag{21}$$

This equation is solved by the substitution

$$\hat{\Delta}_{Br_k}^{(i)} = \hat{G}_{\Delta_k}\hat{\lambda}_k^i \tag{22}$$

where $\hat{G}_{\Delta_k}$ is a column matrix of displacement eigenvectors. In order to simplify the equations, the subscript k is subsequently deleted, until the point in the analysis is reached where it must be reinstated.

$$\therefore \quad [\hat{A}_{Blr}\hat{\lambda}^2 + (\hat{A}_{Bll}+\hat{A}_{Brr})\hat{\lambda} + \hat{A}_{Brl}]\hat{G}_\Delta = 0 \tag{23}$$

Equation (23) is a characteristic equation, but unfortunately it is of the second order in $\hat{\lambda}$. This circumstance constitutes a real difficulty. However, the equation is worth solving, because the matrices are only half as large as the matrices in Eq. (14).

Symmetry and Antisymmetry

Consider the truss substructure of Fig. 3. The stiffness matrix for the substructure is

$$K_B = \left[\begin{array}{cc:cc|cc:cc} \frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -1 & \frac{3}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \hdashline -\frac{1}{2} & 0 & \frac{3}{2} & 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ \hline 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{3}{2} & -1 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -1 & \frac{3}{2} & 0 & -\frac{1}{2} \\ \hdashline 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{3}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} \end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \end{array}\right\} a \\ \left.\begin{array}{l} \\ \end{array}\right\} b \\ \left.\begin{array}{l} \\ \end{array}\right\} a \\ \left.\begin{array}{l} \\ \end{array}\right\} b \end{array} \begin{array}{l} \\ l \\ \\ r \end{array} \tag{24}$$

$$\underbrace{\underbrace{\quad}_{a}\;\underbrace{\quad}_{b}}_{l}\qquad \underbrace{\underbrace{\quad}_{a}\;\underbrace{\quad}_{b}}_{r}$$

This matrix has a certain regularity, based upon the following vector classification: 1) Vectors that are symmetric about the plane of symmetry of the substructure (subsequently called "a" vectors). 2) Vectors that are antisymmetric about the plane of symmetry (subsequently called "b" vectors). Thus Δ_1, Δ_2, Δ_5, and Δ_6 are "a" vectors. The others are "b" vectors. In accordance

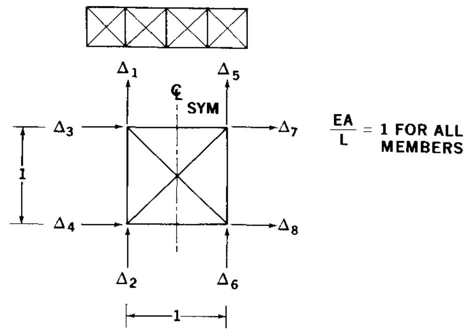


Fig. 3 A truss substructure.

with this classification, the partitions of K_B are seen to conform to the following pattern:

$$K_B = \left[\begin{array}{cc|cc} p_{aa} & p_{ab} & q_{aa} & q_{ab} \\ p_{ab}^T & p_{bb} & -q_{ab}^T & q_{bb} \\ \hline q_{aa} & -q_{ab} & p_{aa} & -p_{ab} \\ q_{ab}^T & q_{bb} & -p_{ab}^T & p_{bb} \end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} l \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} r \end{array} \tag{25}$$

$$\underbrace{\qquad\qquad}_{l}\quad\underbrace{\qquad\qquad}_{r}$$

As a result of the analogy, mentioned previously, between A_B and K_B, that the mechanical impedance matrix can be regarded as a complex stiffness matrix, the impedance matrix follows the same pattern

$$A_B = \left[\begin{array}{cc|cc} \alpha_{aa} & \alpha_{ab} & \mu_{aa} & \mu_{ab} \\ \alpha_{ab}^T & \alpha_{bb} & -\mu_{ab}^T & \mu_{bb} \\ \hline \mu_{aa} & -\mu_{ab} & \alpha_{aa} & -\alpha_{ab} \\ \mu_{ab}^T & \mu_{bb} & -\alpha_{ab}^T & \alpha_{bb} \end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} l \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} r \end{array} \tag{26}$$

$$\underbrace{\qquad\qquad}_{l}\quad\underbrace{\qquad\qquad}_{r}$$

This pattern is of key importance in the further development of the method. The preceding definitions of "a" and "b" vectors apply to force vectors. The classification of a moment vector can be established by considering it as a couple composed of two force vectors. The moment vector has the same classification as the force vectors in the equivalent couple. Figure 4 shows the classification of some vectors acting on a stiffened shell.

Form of the Transformation Matrix

The transformation T is assumed to be of such form that the pattern of A_B defined by Eq. (26) is retained when the partitions of the transformed impedance matrix $\hat{A}_B$ are calculated.

Elimination of Force Variables when $\hat{n}_a = \hat{n}_b$

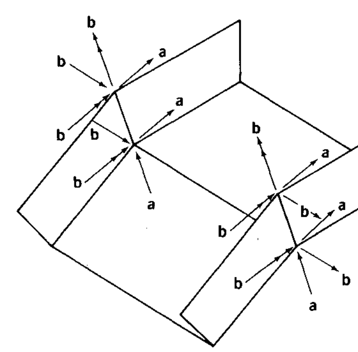
Equation (23) can be written

$$[(\hat{A}_{Blr}+\hat{A}_{Brl})\bar{\lambda} + \hat{A}_{Bll} + \hat{A}_{Brr} + (\hat{A}_{Blr}-\hat{A}_{Brl})\lambda^*]\hat{G}_\Delta = 0 \tag{27}$$

where

$$\bar{\lambda} = \frac{1}{2}\left(\hat{\lambda} + \frac{1}{\hat{\lambda}}\right) \qquad \lambda^* = \frac{1}{2}\left(\hat{\lambda} - \frac{1}{\hat{\lambda}}\right) \tag{28}$$

Fig. 4 *a***-type and** *b***-type vectors.**



Extracting partitions $\hat{A}_{Blr}$, $\hat{A}_{Brl}$, $\hat{A}_{Bll}$, and $\hat{A}_{Brr}$ from $\hat{A}_B$ and substituting into Eq. (27) gives

$$\left.\begin{aligned}(\hat{\mu}_{aa}\bar{\lambda}+\hat{\alpha}_{aa})\hat{G}_{\Delta a}+\hat{\mu}_{ab}\lambda^*\hat{G}_{\Delta b}=0\\ -\mu_{ab}^T\lambda^*\hat{G}_{\Delta a}+(\hat{\mu}_{bb}\bar{\lambda}+\hat{\alpha}_{bb})\hat{G}_{\Delta b}=0\end{aligned}\right\} \tag{29}$$

where $\hat{G}_{\Delta a}$ and $\hat{G}_{\Delta b}$ are column matrices of "a" and "b" displacement eigenvectors. Eliminating $\hat{G}_{\Delta b}$ and λ^* from these equations gives

$$(M_1\bar{\lambda}^2+M_2\bar{\lambda}+M_3)\hat{G}_{\Delta a}=0 \tag{30}$$

where

$$M_1=\hat{\mu}_{ab}^T+\hat{\mu}_{bb}b_{ba} \quad M_2=\hat{\mu}_{bb}a_{ba}+\hat{\alpha}_{bb}b_{ba} \quad M_3=\hat{\alpha}_{bb}a_{ba}-\hat{\mu}_{ab}^T$$
$$a_{ba}=\hat{\mu}_{ab}^{-1}\hat{\alpha}_{aa} \qquad b_{ba}=\hat{\mu}_{ab}^{-1}\hat{\mu}_{aa} \tag{31}$$

Again the characteristic equation is of the second degree in the eigenvalue $\bar{\lambda}$. However, the order of the coefficient matrices is only half as great as the order of the coefficient matrices in Eq. (23). Equation (30) can be solved in the following manner:

Let

$$A_a=\begin{bmatrix}0 & \vdots & I\\ \cdots & \cdot & \cdots\\ -B & \vdots & -A\end{bmatrix} \quad \text{where } \begin{aligned}A&=M_1^{-1}M_2\\ B&=M_1^{-1}M_3\end{aligned} \quad Y_a=\begin{Bmatrix}\hat{G}_{\Delta a}\\ Z\end{Bmatrix} \tag{32}$$

Then the equation

$$(A_a-\bar{\lambda}I)Y_a=0 \tag{33}$$

can be shown to be equivalent to Eq. (30). The partition Z of Y_a is equal to $\bar{\lambda}\hat{G}_{\Delta a}$. This partition is introduced to aid the solution, and is subsequently discarded. The order of A_a is equal to $2\hat{n}_a=\hat{n}$. Thus force variables have been eliminated and the matrix A_a is half the order of matrix $\hat{N}$ in Eq. (14).

From Eq. (28)

$$\hat{\lambda}=\bar{\lambda}\pm(\bar{\lambda}^2-1)^{1/2} \tag{34}$$

From Eq. (29)

$$\hat{H}_{\Delta b}=-(b_{ba}\hat{H}_{\Delta a}\bar{\lambda}_D+a_{ba}\hat{H}_{\Delta a})(\lambda_D^*)^{-1} \tag{35}$$

where λ^* is obtained from Eqs. (28) and (34), $\hat{H}_{\Delta a}$ and $\hat{H}_{\Delta b}$ are rectangular matrices of "a" and "b" type displacement eigenvectors $\hat{G}_{\Delta a_k}$ and $\hat{G}_{\Delta b_k}$, and $\bar{\lambda}_D$ and λ_D^* are diagonal matrices of eigenvalues $\bar{\lambda}_k$ and λ_k^*.

The force eigenvectors can be obtained from Eq. (19). Let

$$\hat{P}_{UBr}^{(i)}=\hat{G}_P\hat{\lambda}^i \tag{36}$$

where $\hat{G}_P$ is a column matrix of force eigenvectors. Substituting $\hat{P}_{UBr}^{(i)}$ from Eq. (36) and $\hat{\Delta}_{Br}^{(i)}$ from Eq. (22) into Eq. (19) after the superscripts have been increased by unity gives

$$\hat{G}_{P_k}=-\hat{A}_{Bll}\hat{G}_{\Delta_k}-\hat{A}_{Blr}\hat{G}_{\Delta_k}\hat{\lambda}_k \tag{37}$$

$$\therefore\ \hat{H}_P=-\hat{A}_{Bll}\hat{H}_\Delta-\hat{A}_{Blr}\hat{H}_\Delta\hat{\lambda}_D \tag{38}$$

where $\hat{H}_P$ and $\hat{H}_\Delta$ are square matrices of force and displacement eigenvectors $\hat{G}_{P_k}$ and $\hat{G}_{\Delta_k}$, and $\hat{\lambda}_D$ is a diagonal matrix of eigenvalues $\hat{\lambda}_k$.

Nature of the Roots

From Eq. (34), the product of the two roots is

$$\hat{\lambda}_1\hat{\lambda}_2=1 \tag{39}$$

Thus the reciprocal of every eigenvalue is an eigenvalue. From Eqs. (16) and (39), the following results can be deduced:

$$\hat{r}_1\hat{r}_2=1 \qquad \hat{\theta}_1+\hat{\theta}_2=2k\pi \qquad k=0,1,2,\ldots \tag{40}$$

Thus the reciprocal of the modulus of every eigenvalue is the modulus of another eigenvalue.

Equation (17) then shows that for every function that increases to the right, in the direction of increasing i (i.e., for $\hat{r}>1$), a similar function exists that increases to the left (for $\hat{r}<1$). Thus a result which has previously been shown to apply to other classes of structures, such as the rib-skin structures of Ref. 6, is demonstrated to be applicable to the general class of damped periodic structures under consideration.

Whenever damping is present, which is always the case for real structures, all of the roots $\hat{\lambda}$ are complex and distinct. The responses corresponding to these roots are exponentially damped waves in the longitudinal direction.

When damping is zero, the $\hat{N}$ matrix [Eq. (12)] is real and the roots $\hat{\lambda}$ are real or complex conjugate. The modulus of the complex conjugate roots $\hat{r}$ can be unity or different from unity. When $\hat{r}$ is unity, the response is an undamped sinusoidal wave. Otherwise the wave is damped. When the roots are real, the response varies exponentially in the longitudinal direction. In the vibration problem all the roots are distinct. When the MDE method is applied to the stability or static stress problems, real repeated roots equal to unity can occur. In this case the responses are algebraic functions of the longitudinal coordinate.

In the vibration of undamped infinitely long structures, natural modes and frequencies can occur, as pointed out by Mead.[7] The term "natural frequency" in the present context refers to a frequency at which a finite forcing function produces infinite response. It can be shown that these modes and frequencies can be calculated by setting $\hat{\lambda}$ equal to plus or minus unity in Eq. (23). The result is a characteristic equation in the eigenvalue ω, which is contained in the partitions of the impedance matrix, and in the eigenvectors $\hat{G}_\Delta$. When $\hat{\lambda}$ is set equal to ± 1, it will be found that Eq. (23) decomposes into two characteristic equations, one involving only a-type displacements, and the other involving only b-type displacements, as a result of the form of the partitioned impedance matrix, Eq. (26). Thus, four kinds of natural modes exist, designated Types 1-a, 1-b, 2-a, and 2-b. Type 1-a modes involve only a-type displacements, while Type 1-b modes involve only b-type displacements. Types 1-a and 1-b correspond to $\hat{\lambda}=1$; consequently, displacements at all stations are identical. Type 2-a (a-type displacements) and type 2-b (b-type displacements) also exist. These modes correspond to $\hat{\lambda}=-1$; consequently, displacements at successive stations are equal and opposite. The discovery of the existence of a-type and b-type natural modes exemplifies the added insight resulting from the present approach to the problem of vibrating periodic structures.

Force Variable Elimination for $\hat{n}_a>\hat{n}_b$ and $\hat{n}_a<\hat{n}_b$

The solutions for these cases are beyond the scope of the present paper, but the derivations follow the lines of the derivation for $\hat{n}_a=\hat{n}_b$, and the results are similar.

Boundary Conditions

The MDE method is applicable to bounded (finite length) and unbounded (infinite length) structures, as mentioned in a preceding paragraph. Both cases have been analyzed, and computer programs based upon these analyses are in use. The bounded structure analysis allows the user to specify a condition of zero displacement or zero force in each structural element at both ends of the structure.

Unbounded Structure

Because of space limitations, only the equations for this case are presented. As Eq. (40) shows, one-half of all eigenvalues have moduli greater than unity. These eigenvalues, $\hat{n}$ in number, are not required, since the response of the structure as i approaches infinity is zero. The use of these eigenvalues in calculating response is implied through considerations of symmetry and antisymmetry.

The only other boundary conditions considered are external loading conditions at $i=0$. The case of a uniform structure loaded at a finite number of stations can be covered by superimposing solutions for structures loaded at $i=0$.

Loads at $i=0$

The loading conditions are divided into two categories: symmetric and antisymmetric. In the symmetric conditions a-type loads are specified, and b-type displacements are zero. For example, if an external radial load only is acting at a station, then longitudinal displacements and rotations about radial and tangential axes (i.e., tangent to the circumference of the cross section) are zero. In the antisymmetric conditions b-type loads are specified, and a-type displacements are zero.

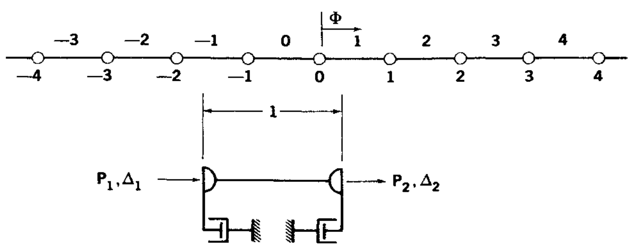


Fig. 5 A simple structure and a substructure.

Let Φ_a = a column matrix of complex amplitudes of a-type loads externally applied at $i = 0$. The real and imaginary parts of Φ_a define the loads in the usual manner. For the symmetric case

$$P^{(1)}_{UBla} = -P^{(0)}_{UBra} = \tfrac{1}{2}\Phi_a \qquad \Delta^{(0)}_{Brb} = 0 \tag{41}$$

where $P^{(0)}_{UBra}$ is a partition of $P^{(0)}_{UBr}$ containing a-type forces. Multiplying $P^{(0)}_{UBra}$ in Eq. (41) on the left by the applicable partitions of the transformation matrix T gives

$$\hat{P}^{(0)}_{UBra} = -\tfrac{1}{2}\hat{\Phi}_a \tag{42}$$

The condition involving $\Delta^{(0)}_{Brb}$ in Eq. (41) is satisfied if

$$\hat{\Delta}^{(0)}_{Brb} = 0 \tag{43}$$

From Eqs. (12) and (18)

$$\hat{P}^{(0)}_{UBra} = \hat{H}_{Pa} C \qquad \hat{\Delta}^{(0)}_{Brb} = \hat{H}_{\Delta b} C \tag{44}$$

where C is a column matrix of arbitrary constants. The matrix $\hat{H}_{Pa}$ can be obtained from $\hat{H}_P$, Eq. (38). From Eqs. (42), (43), and (44)

$$B_S C_S = \Psi_S \tag{45}$$

where

$$B_S = \begin{bmatrix} \hat{H}_{Pa} \\ \hat{H}_{\Delta b} \end{bmatrix} \qquad \Psi_S = \begin{Bmatrix} -\tfrac{1}{2}\hat{\Phi}_a \\ 0 \end{Bmatrix} \tag{46}$$

and the subscript S denotes the symmetric case. Equation (45) can be solved for C_S. Arbitrary constants for the antisymmetric case can be derived in a similar manner.

Applications

Simple Example

Figure 5 shows a simple structure composed of axially loaded bars, masses, and dampers. The structure is infinitely long, and an oscillatory force Φ is acting at $i = 0$, such that $\Phi_R = -1$, $\Phi_I = 0$. The frequency is $\omega = 1$. The substructure diagram shows dimensions and the numbering of forces and displacements. The stiffness, mass, and damping matrices are

$$K_B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad M_B = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \qquad C_B = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

From Eqs. (4), (8), (27), (34), (37), and (16), in that order

$$\hat{G} = \begin{Bmatrix} j(2)^{1/2} \\ 1 \end{Bmatrix}, \begin{Bmatrix} -j(2)^{1/2} \\ 1 \end{Bmatrix}$$

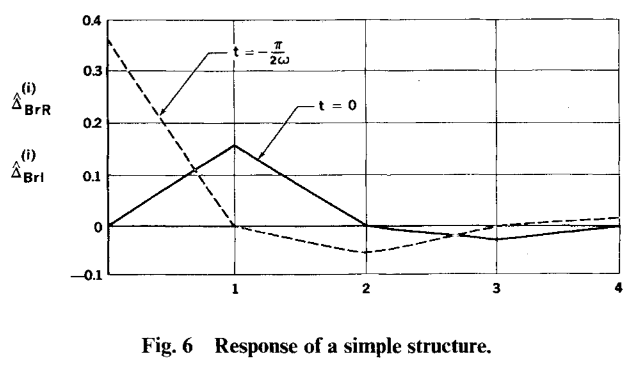


Fig. 6 Response of a simple structure.

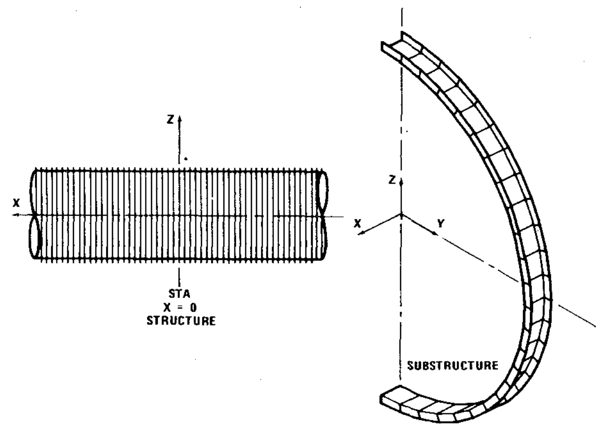


Fig. 7 Engine duct finite element model.

$$\hat{r}_1 = 1 + (2)^{1/2} \qquad \hat{r}_2 = -1 + (2)^{1/2}$$

$$\hat{\theta}_1 = \frac{\pi}{2} \qquad \hat{\theta}_2 = -\frac{\pi}{2}$$

From Eq. (17)

$$\hat{Y}^{(i)} = C_1 \begin{Bmatrix} j(2)^{1/2} \\ 1 \end{Bmatrix} [1 + (2)^{1/2}]^i \left(\cos\frac{\pi}{2} i + j \sin\frac{\pi}{2} i \right) + C_2 \begin{Bmatrix} -j(2)^{1/2} \\ 1 \end{Bmatrix} [-1 + (2)^{1/2}]^i \left(\cos\frac{\pi}{2} i - j \sin\frac{\pi}{2} i \right)$$

Boundary conditions: As $i \to \infty$, $\hat{Y}^{(i)} \to 0$. Therefore, discard the first term of the above expression since $1 + (2)^{1/2} > 1$. At

$$i = 0, \quad \hat{P}^{(1)}_{UBl} = -\tfrac{1}{2} = -\hat{P}^{(0)}_{UBr} \quad \therefore \; C_2 = \frac{j}{2(2)^{1/2}}$$

$$\therefore \; \hat{P}^{(i)}_{UBr} = \tfrac{1}{2}[-1 + (2)^{1/2}]^i \left(\cos\frac{\pi}{2} i - j \sin\frac{\pi}{2} i \right)$$

$$\hat{\Delta}^{(i)}_{Br} = \frac{(2)^{1/2}}{4}[-1 + (2)^{1/2}]^i \left(\sin\frac{\pi}{2} i + j \cos\frac{\pi}{2} i \right)$$

Figure 6 shows the variation of $\hat{\Delta}^{(i)}_{Br}$ with i. Note that the response has the character of a traveling wave.

Application to Acoustic Vibration

An aircraft engine duct was analyzed as a means of demonstrating the capabilities of the MDE approach. Calculations were accomplished with the aid of a FORTRAN computer program designed for the purpose. Figure 7 is a diagram of the duct finite element model. The structure is a circular cylindrical aluminum alloy shell about 7 ft in diam, stiffened by exterior circumferential frames $5\frac{1}{2}$ in. apart. The actual duct is about 24 ft long. The model is infinitely long. Figure 7 also shows the substructure modeled with shear panels and bar elements that carry

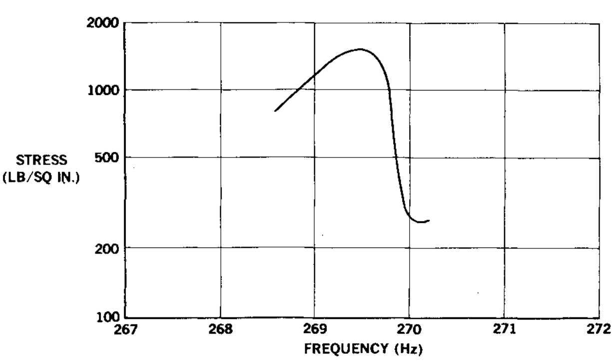


Fig. 8 Resonance peak for the vibrating duct.

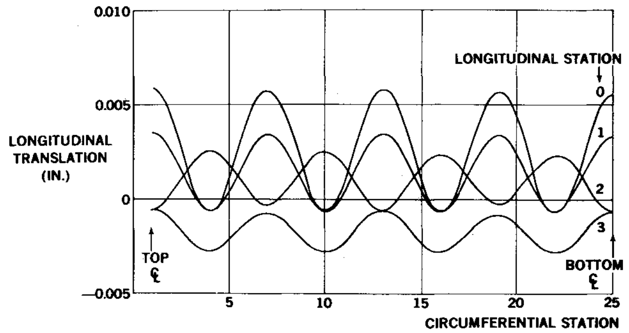


Fig. 9 Displacement of frame flanges.

axial load, bending and shear. Stiffness, mass, and damping matrices were calculated for the substructure. The only damping considered was air damping. The exciting force was engine noise on the duct interior. The force was assumed to be a pressure distribution that varied sinusoidally in the circumferential and longitudinal directions, constituting a traveling wave.

The infinite model has natural frequencies that produce resonance peaks in the frequency-response curve. Figure 8 shows such a peak at a frequency in the neighborhood of 269 Hz, as calculated by the MDE program. The response plotted is the rms value of bending stresses in frame webs about axes tangent to the shell circumference at station zero. When such a peak is located, details of the response are calculated by a postprocessor computer program. In this case detailed responses were output at 26 locations along the longitudinal axis. Figure 9 shows longitudinal displacement of the outer frame flange at four adjacent stations at $t = 0$. Figure 10 shows frame web bending stresses at station zero at $t = 0$, and a quarter of a cycle earlier. Stresses are given at 52 points around the circumference. Figure 11 shows the variation along the length of the duct at the point of maximum stress (where there is a frame splice), at $t = 0$ and a quarter of a cycle earlier. The response has the character of a traveling wave.

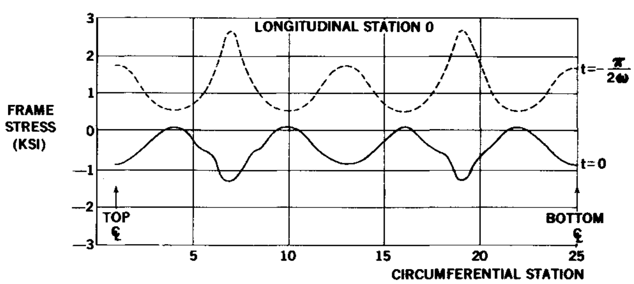


Fig. 10 Circumferential frame stress distribution.

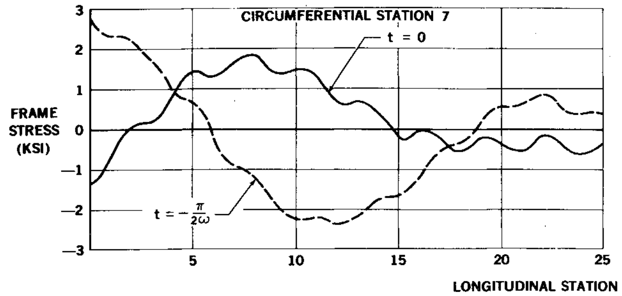


Fig. 11 Longitudinal frame stress distribution.

Conclusions

The general MDE approach to the analysis of periodic structures has been presented. The mathematical formulation, suitably implemented, provides a capability to perform vibration analyses of periodic structures. The computing effort required to analyze a finite element model composed of many substructures (even infinitely many) is on the order of the effort required to analyze a single substructure. An improved capability for period structural analysis is thus provided.

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